Fujita approximations

X irreducible, projective, dim n.

Let $\xi \in N^1(X)_{\mathbb{R}}$ be big. A *Fujita approximation* for ξ is a birational morphism $\mu: X' \to X$, X' irreducible, and a decomposition

$$\mu^*(\xi) = \underset{\substack{\uparrow \\ \text{ample}}}{a} + \underset{\substack{\uparrow \\ \text{effective}}}{e} \qquad (a, e \in N^1(X')_{\mathbb{R}})$$

Rmk: Given a decomp. as above, we get

$$\mathrm{vol}_X(\xi) = \mathrm{vol}_{X'}(\mu^* \xi) \geq \mathrm{vol}_{X'}(a) = a^n.$$

In fact, we can choose a Fujita approx so that the volume of ξ is arbitrarily close to a^n:

Fujita's approximation Theorem: Let $\xi \in N^1(X)_{\mathbb{R}}$ be big. Fix $\varepsilon > 0$. Then there exists a Fujita approximation

$$\mu: X' \to X, \quad \mu^*(\xi) = a + e$$

s.t. $\mathrm{vol}_{X'}(a) > \mathrm{vol}_X(\xi) - \varepsilon$

(i.e. $a^n + \varepsilon > \mathrm{vol}(\xi) \geq a^n$).

Additionally, if ξ is in $N^1(X)_{\mathbb{Q}}$, then can take $a, e \in N^1(X)_{\mathbb{Q}}$

The corresponding approximation is called a *Fujita ε-approximation of ξ.*

(Pf relies on multiplier ideals — see chapter 11 of Pos. II)

<u>Side note</u>: If D is a divisor on X, then D is big and nef $\Longleftrightarrow$ $\exists$ effective N s.t. $D - \frac{1}{k}N$ is ample for $k >> 0$.

<u>Pf</u>: For "$\Leftarrow$": Bigness is clear. Nefness follows from being a limit of ample classes

For "$\Rightarrow$": D big $\Rightarrow$ $mD \equiv \underset{\text{ample}}{A} + \underset{\text{effective}}{N}$ for some $m > 0$.

$\Rightarrow$ for $k > m$, $kD \equiv \underbrace{\big((m-k)D + A\big)}_{\text{ample}} + N \Rightarrow D - \frac{1}{k}N$ is ample. $\square$

<u>Rmk</u>: Let $\mu : X' \to X$, $\mu^*(\xi) = a + e$ be a Fujita ε-approximation, ξ big.

Let $\nu : X'' \to X'$ be a projective, birational morphism.

Then $\nu^*(a)$ is big and nef.

$\Rightarrow$ Can find arb. small effective e_0 on X'' s.t.

$$a_1 = \nu^*(a) - e_0 \quad \text{is ample. (by above)}$$

Set $e_1 = \underbrace{\nu^*(e) + e_0}_{\text{effective}}$, and $\mu_1 = \mu \circ \nu$

Then $\mu_1^*(\xi) = (a_1 + e_0) + (e_1 - e_0) = a_1 + e_1$

So μ_1 also gives a Fujita approximation.

Moreover, $\operatorname{vol}_{X'}(a) > \operatorname{vol}_X(\xi) - \varepsilon$

$$\operatorname{vol}_{X'}(a) = \operatorname{vol}_{X''}(a_1 + e_0)$$

By continuity of volume, we can choose e_0 suff. small so that

$$\operatorname{vol}_{X''}(a_1) > \operatorname{vol}_X(\xi) - \varepsilon.$$

i.e. μ_1 gives an ε-approx. as well

$\Rightarrow$ in the theorem we can assume X' is smooth.

<u>Rmk</u>: Suppose L is big and integral on X and fix $\varepsilon > 0$.

Then the theorem implies $\exists$ $\mu : X' \to X$ birational (X' smooth)

and $\mu^*(pL) = \underset{\text{ample}}{\overset{pa}{A}} + \underset{\text{effective}}{\overset{pe}{E}}$, A, E integral

and $\operatorname{vol}(a) > \operatorname{vol}(L) - \varepsilon$

i.e. $A^n = \operatorname{vol}(A) > p^n(\operatorname{vol}(L) - \varepsilon)$

<u>Cor</u> (of thm): Let L be an integral divisor on X.

Then $\operatorname{vol}(L) = \liminf_{m\to\infty} \dfrac{h^0(mL)}{m^n/n!}$

That is, $\operatorname{vol}(L)$ is actually the <u>limit</u> of the corresponding sequence.

Pf: obvious if L isn't big, so assume L is big.

By a previous argument, we can assume X is normal

(i.e. take L' = pullback along normalization. $h^0(mL)$ will grow at same rate as $h^0(mL')$.)

Fix ε and let $\mu: X' \to X$, $\mu^*(pL) \equiv_{\text{lin}} A + E$ be the Fujita approx. given in the previous remark, w/ X' normal.

L big $\Rightarrow \exists\, q_0$ s.t. rL is effective for $r \in [q_0 p, (q_0+1)p]$.

So $h^0(\mathcal{O}_X((kp+r)L)) \geq h^0(\mathcal{O}_X(kpL)) \geq h^0(\mathcal{O}_{X'}(kA)) \quad \forall\, k \geq 0.$

By Asymptotic R-R, since A is ample,

$$h^0(kA) = \frac{A^n}{n!} k^n + O(k^{n-1}) \geq \frac{A^n}{n!} k^n - (p^n \varepsilon) \cdot \frac{k^n}{n!} \quad \text{for } k \gg 0.$$
$$= (A^n - p^n \varepsilon) \frac{k^n}{n!}$$
$$> \left(p^n (\mathrm{vol}(L) - 2\varepsilon)\right) \frac{k^n}{n!}$$

$$\Rightarrow \frac{h^0((kp+r)L)}{(kp)^n / n!} > \mathrm{vol}(L) - 2\varepsilon \quad \text{for } k \gg 0$$

$$\Rightarrow \liminf_{m \to \infty} \frac{h^0(mL)}{m^n / n!} > \mathrm{vol}(L) - 2\varepsilon \quad \text{for any } \varepsilon > 0. \quad \square$$

Apr 14

Another way to calculate volume involves finding a more general analogue of self-intersection numbers:

If A is ample, then we know $\mathrm{vol}(A) = A^n$.

Geometrically, if we choose $m>0$ s.t. mA is v. ample,

Then for general $D_1, D_2, \ldots, D_n \in |mA|$,

$$\mathrm{vol}(A) = \frac{1}{m^n} \# (D_1 \cap D_2 \cap \ldots \cap D_n)$$

More generally, let L be a big divisor on X.

Fix $m>0$ suff. large so that $|mL|$ defines a birational map from X to its image.

Denote $B_m = Bs(|mL|)$.

Choose n general divisors $D_1, \ldots, D_n \in |mL|$

Then the <u>moving self intersection number</u> of $|mL|$ is

$$(mL)^{[n]} := \# (D_1 \cap \ldots \cap D_n \cap (X - B_m))$$

<u>Thm</u>: Let L be a big divisor on a normal projective variety X. Then

$$\mathrm{vol}(L) = \limsup_{m \to \infty} \frac{(mL)^{[n]}}{m^n}$$

Pf: Let $\nu_m : X_m \longrightarrow X$ be the resolution of the blowing up of X along the base ideal $b(|mL|)$.

Then $\nu_m^*(|mL|) = |P_m| + F_m$, where F_m is the fixed locus.

Projection formula $\Rightarrow$ $\nu_{m*}\nu_m^*(mL) = \nu_{m*}\mathcal{O}_{X'} \otimes \mathcal{O}_X(mL)$ ($\mathcal{O}_X$ since X is normal)

so $H^0(\mathcal{O}_X(mL)) = H^0(\mathcal{O}_{X_m}(\nu^*(mL))) = H^0(\mathcal{O}_{X_m}(P_m))$

i.e. $\nu_m^*(|mL|) = |\nu_m^*(mL)|$ and

$P_m := \nu_m^*(mL) - F_m$ is basepoint free.

Thus, $(mL)^{[n]}$ = # intersection pts of n divisors in $|P_m|$.
$= (P_m)^n$

P_m is globally generated and thus nef, so $(P_m)^n = \mathrm{vol}(P_m)$

$$\underset{\substack{\parallel \\ m^n \mathrm{vol}(L)}}{\mathrm{vol}(mL)} = \mathrm{vol}(\mu^*(mL)) \geq \mathrm{vol}(P_m) = (mL)^{[n]} \quad \text{for } m \gg 0.$$

$\Rightarrow$ $\mathrm{vol}_X(L) \geq \dfrac{(mL)^{[n]}}{m^n}$ for $m \gg 0$.

Now we want to show there is some <u>subsequence</u> that is eventually within ε of $\mathrm{vol}(L)$.

Fix $\varepsilon > 0$.

Find an ε-approximation:

$$\mu : X' \longrightarrow X \ , \ \mu^* L = \underset{\substack{\uparrow \\ \text{ample}}}{A} + \underset{\substack{\uparrow \\ \text{eff.}}}{E} \ ,$$ where A, E are $\mathbb{Q}$-divisors

Choose $k \in \mathbb{Z}_{>0}$ s.t. kA is integral and globally generated.

Now consider birational map $\nu_k : X_k \to X$

so that $\nu_k^*(kL) \equiv_{lin} P_k + F_k$.

WLOG, can assume $X_k = X'$ (pass to a common resolution)

So $\nu_k^*(kL) \equiv_{lin} A_k + E_k$

$$A_k = \underset{\substack{\uparrow \\ \text{glob. gen}}}{kA} \qquad E_k = \underset{\substack{\uparrow \\ \text{effective}}}{kE}$$

and $(A_k)^n \geq k^n(\mathrm{vol}(L) - \varepsilon)$

$|A_k|$ is b.p.f., and F_k is the fixed locus of $\nu_k^*(kL)$, so

$$P_k + F_k \equiv_{lin} A_k + E_k \equiv A_k + \text{effective} + F_k$$

$$\Rightarrow A_k^n \leq P_k^n = (kL)^{[n]}$$

$$\frac{(kL)^n}{k^n} \geq \mathrm{vol}(L) - \varepsilon$$ for $k \gg 0$, sufficiently divisible.

Thus, $\mathrm{vol}(L) = \limsup\limits_{m \to \infty} \dfrac{(mL)^{[n]}}{m^n}$ as desired. $\square$